PROPAGATION OF THE FRONT OF A STRONG SHOCK WAVE IN AN INHOMOGENEOUS ATMOSPHERE

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An equation describing the propagation of the front of a point explosion in an inhomogeneous atmosphere is derived here with the aid of dimensional analysis. The solution for an exponential atmosphere agrees with already known results.

The problem concerning the propagation of the front of a strong shock wave in an inhomogeneous medium, as is well known, is not an autonomous one. For a power-law density distribution relative to the center

$$\rho = ar^l;\ a, l = \text{const}, \tag{1}$$

the method by L. I. Sedov [1] reduces the problem to the fundamental autonomous problem of a strong explosion in a homogeneous medium. In that problem the front propagation R(t) is determined without regard to the distribution of gasodynamic parameters behind the front. More precisely, the law of motion is determined for the wave front within an accuracy down to a constant (of the order of unity) calculated from the energy integral.

For an exponential-law atmosphere, A. S. Kompaneets [2] has developed an approximate method in which he assumed the gas mass to be concentrated primarily near the front (within a layer of thickness $\Delta \sim (\gamma - 1/\gamma + 1)R$, tending toward zero as $\gamma \to 1$). During an upward motion ($\theta = 0$), toward lower densities, the front is accelerated fast and within a relatively short time interval (approximately one second for energies of the order of 10^{22} ergs) it approaches infinity; for a downward motion this method is unsuitable, on the other hand, even though the shock wave remains strong for a few tenths of a second. Not only is it difficult to numerically integrate partial differential equations in coordinates (θ, t), as has been shown by experience, but the results do not apply to the real nonexponential atmosphere of the Earth. On account of the fast increasing value of the inhomogeneity parameter $h(\theta)$ as a function of the explosion altitude H, for instance, the time of atmospheric breakdown $t_0 \sim (\rho_H h_H^5)^{1/2}$ would increase with the altitude, which is absurd.

In order to describe how the front of a strong shock wave propagates after a downward breakdown ($\theta = \pi$) in an exponential-law atmosphere, Yu. P. Raizer [3] has proposed an interpolation between his autonomous solution for a plane shock at infinity and the solution by A. S. Kompaneets at an instant near breakdown. Unfortunately, this interpolation method is not satisfactory and its application alone is fraught with difficulties (details will be discussed later).

The author proposes an approximate method of describing the propagation of a point explosion in an exponential-law and in the real atmosphere, which would match those two solutions within the respective intervals.

Generally, when the density ρ is referred to a dimension of length h, the velocity (as well as any other gasodynamic quantity) can be expressed in terms of two variables (r/R, r/h) rather than in terms of (r, t). The velocity at the front (r = R) will be then

$$u = \dot{R} = \frac{R}{t}\varphi_1(R/h) + \frac{h}{t}\varphi_2(R/h). \tag{2}$$

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With respect to the dimensionless variable

$$x = R/h(\theta) \tag{3}$$

we obtain

$$\dot{x} = \frac{1}{t}\Phi(x). \tag{4}$$

This phenomenological equation will be our point of departure. It is easy to find the limit values of function $\Phi(x)$:

$$\Phi(x) \to \text{const}\, x = k_1 k_2 x \quad \text{at} \quad x \to 0 \ (h \to \infty),$$

corresponding to the transition to a homogeneous atmosphere, and

$$\Phi(x) = \text{const} = k_1 \quad \text{at} \quad x \to \infty \ (h \to 0),$$

corresponding to the limiting case of motion in an inhomogeneous atmosphere (toward higher densities). The latter expression matches the earlier mentioned autonomous solution [3].

Since the density at the wave front depends on $x = R/h$, hence $\Phi(x)$ can be expressed in terms of density, which determines the characteristics of the motion. In the phenomenological method it is desirable to represent the unknown function $\Phi(x)$ in terms of a few parameters, and these should be determined from any known data (experiment, solutions at the limits). Through the said conditions at the limits, function $\Phi(x)$ can be represented by the following simpler (power-law) relation:

$$\Phi(x) = k_1\left[1 - (\rho_H/\rho)^{k_2}\right], \tag{5}$$

with ρ_H denoting the density at the altitude (point) of explosion. Specifically, for an exponential-law atmosphere

$$\rho/\rho_H = e^{R/h(\theta)}, \quad h(\theta) = h_\pi/(1 - \cos\theta). \tag{6}$$

With the aid of (5), the equation of motion (4) can be rewritten as

$$\dot{x} = \frac{k_1}{t}\left[1 - (\rho_H/\rho)^{k_2}\right]. \tag{7}$$

For an exponential-law atmosphere (6) the front propagation $R(\theta, t)$ is expressed in terms of x as follows:

$$\int_{-\infty}^{x} (1 - e^{-k_2 x})^{-1}\,dx = k_1 \ln t. \tag{8}$$

At a constant h (for an exponential-law atmosphere) it is worthwhile to introduce the time-scale factor

$$\tau = (\rho_H h_\pi^5/\xi E)^{1/2}. \tag{9}$$

Equation (7) and solution (8) do not change as a result. For small values of t (or x) solution (8) should become the Sedov solution for a homogeneous atmosphere, in dimensionless variables

$$x = t_1^{2/5}; \qquad t_1 = t/\tau. \tag{10}$$

A comparison with the limit condition derived from (8) yields

$$k_1 k_2 = 2/5. \tag{11}$$

The second equation for determining the constants k_1 and k_2 will be obtained from the condition that solution (8) become the autonomous solution in [3] for $\theta = \pi$ (downward motion). For $x > 0$ we obtain from (8)

$$\int_{0}^{x} (1 - e^{-k_2 x})^{-1}dx = k_1 \ln\frac{t}{t_0}, \tag{12}$$

where

$$t_0 = \exp\left[\frac{1}{k_1}\int_{-\infty}^{0} (1 - e^{-k_2 x})^{-1}dx\right] \tag{13}$$

has the meaning of breakdown time.

For $\theta = \pi$ at the limit $x \to \infty$ ($R \to \infty$) we have from (12)

$$x_\pi = \frac{R}{h_\pi} = k_1 \ln \frac{t}{t_0}. \tag{14}$$

It follows from here that the constant k_1 coincides exactly with the autonomicity criterion in [3] (coefficient of the logarithm) and thus $k_1 = \alpha_p(\gamma)$. For real values of γ (1 to 2) α_p varies from 1 to 3/2. Correspondingly, k_2 varies from 0.4 to 0.25. With $\gamma = 1.25$, for example, we have $\alpha_p = 1.345$ and $k_2 = 0.3$.

A result close to this can be obtained in analytic form, on the basis of Eq. (4) written as

$$\dot{x} = \frac{1}{t} \frac{R}{h} F(x). \tag{15}$$

We will express R in the coefficient here approximately. We note, to begin with, that in the combination of length dimensions [1]

$$R = \text{const} \, (Et^2/\rho)^{2/(\nu+2)} \tag{16}$$

the density of the medium can be treated as a function of the front radius, i.e., $\rho = \rho(R)$, which corresponds to a description of the front propagation in terms of quantities E, t, γ, and one gasodynamic function, in accordance with the earlier discussion. For instance, inserting the power-law expression for the density (1) and solving Eq. (16) for R leads exactly to the expression which L. I. Sedov has obtained by a direct combination of dimensional parameters.

It is possible to show that Eq. (15) with F(x) = const and R replaced by (16) describes the front propagation, within a 30-40% accuracy, and constitutes a complete qualitative representation of the pattern of front propagation in an inhomogeneous atmosphere. The proper choice of the supplementary function F(x) will yield much more precise results. Thus, we will replace R in Eq. (15) by its said approximation. We will, furthermore, express F(x) in terms of $\rho(x)$ according to the conditions at the limits and by again using the power-law relation. Equation (15) becomes then

$$\frac{dx}{ds} = (\rho_H/\rho)^k, \tag{17}$$

with

$$s = (-\cos\theta)\, t_1^{2/5}; \qquad t_1 = t/\tau. \tag{18}$$

With the following designations for an exponential-law atmosphere

$$k = t_{10}^{-2/5} = s_0^{-1}; \qquad t_{10} = k^{-5/2}, \tag{19}$$

we obtain the following solution (in ordinary coordinates)

$$R = \frac{h(\theta)}{k} \ln \left[1 - \cos\theta \left(\frac{t_1}{t_{10}}\right)^{2/5}\right]. \tag{20}$$

From the condition at the limit when $\theta = \pi$ we have

$$x_\pi = \frac{R}{h_\pi} = \frac{1}{k} \frac{2}{5} \ln (t_1/t_{10}). \tag{21}$$

A comparison with the autonomous solution by Yu. P. Raizer (see the discussion following Eq. (14)) yields

$$\frac{2}{5k} = \alpha_p; \qquad k = \frac{2}{5\alpha_p} = \frac{2}{5k_1} = k_2. \tag{22}$$

From (21) we find the breakdown time (in τ-units):

$$t_{10} = t_0/\tau = k^{-5/2} = (5\alpha_p/2)^{5/2}. \tag{23}$$

At $\gamma = 1.25$, for instance, we obtain $t_0 \approx 20\tau$, which is almost identical to the result of calculations by Eq. (13).

Expression (20) becomes exactly the Sedov solution when $t \to 0$ ($t \ll t_0$) and when $h \to \infty$, which corresponds to the transition to a homogeneous atmosphere. When $\theta = 0$ and $t \to t_0$,

$$x \to \frac{5}{2}\alpha_p \ln\left[\frac{2}{5}\left(1-\frac{t}{t_0}\right)\right]. \qquad (24)$$

This expression corresponds to the autonomous Raizer solution for propagation toward lower densities, but the autonomicity criterion is here 1.5 times larger than the one which (24) would yield. By the way, the upward motion following a point explosion is never described by an autonomous solution. We note that no time parameter appears in the autonomous solution. This parameter appears only in the solution by the interpolation method.

It has been noted earlier that this solution coincides with known autonomous solutions on the basis of which the phenomenological parameters are chosen. We will compare the results of solution (8) or (20) with the results of the Kompaneets method [4]. The breakdown times agree almost exactly: $t_0 = 24\tau$, with τ_0 expressed (like τ) according to (9) but without the Sedov-theory parameter ξ. Insufficient data obtained in [4] make it impossible to carry on a precise comparison. However, the characteristic parameters derived here agree closely: the front radius at the instant of breakdown during a downward motion $R(\pi, t_0) \approx 2h_\pi$ and during a horizontal motion $R(\pi/2, t_0) \approx 3h_\pi$, the radius at which the upward velocity is minimum $R(\theta = 0, t^*) = 3h_\pi$, and that minimum velocity. Thus, the agreement of the basic results justifies a conclusion that our approximate method concurs with the Kompaneets method. We note, by the way, that the accuracy of the Kompaneets method for real values of γ is about 20-30%.

We note further that the method proposed by Yu. P. Raizer of interpolating between his autonomous solution and the Kompaneets solution at $t^* \approx 20\tau_0$, near the breakdown time, is not applicable to the velocity, because in the autonomous solution the latter is determined entirely by the parameter α_p and differs appreciably from the velocity in the Kompaneets solution at that instant of time (because transition to the autonomous mode occurs later). Interpolation is feasible and follows naturally along the coordinate which contains the shock parameter A ($g \cdot cm^{-2} \cdot sec^{-1}$) representing the characteristics of prior motion. As a result, we have

$$R = xh_\pi = R(t_0) + \alpha_p h_\pi \ln\left[t/(\rho_1 h_\pi/A)^{1/\alpha_p}\right]. \qquad (25)$$

Here ρ_1 denotes the density at the front at the instant for which interpolation is performed. If the shock parameter is related to the shock mass, then expression (25) will yield the limit condition (21) for the solution obtained here. We note that the shock mass $M(t^*)$ is never described by an autonomous solution, while the mass $M < \bar{M} = 18M^*$ is described only approximately. In an exponential-law atmosphere, a shock wave cannot be considered sufficiently strong any more at a distance corresponding to mass $\bar{M}$ and, therefore, an autonomous solution itself becomes inapplicable. Since the velocity of autonomous propagation does not depend on the energy (it is determined by the parameter $\alpha_p(\gamma)$ only), hence an increase in energy will not extend the applicability of the autonomous solution.

Equation (7) can be used also for determining the front propagation in any inhomogeneous atmosphere characterized by some effective parameter h with the dimension of length. The solution is given by a quadrature expression like (8). Parameters k_1 and k_2 can be selected on the basis of the same concepts, considering the transition to an exponential-law atmosphere.

The density of the Earth's atmosphere can be represented (within measuring accuracy and diurnal fluctuation) by the following formula:

$$\rho(z) = \rho(0)\, e^{-z/h(z)}; \quad h(z) = 7 + 0.026(z - 100), \quad z < 2 \cdot 10^3 \text{ km} \qquad (26)$$

(at $z < 100$ km one may assume $h = 7$). The parameter h introduced here differs from the one based on the method of smoothest tangency and is usually given in the literature. Representation (26) is a point representation and gradually transforms into an exponential-law representation of the atmosphere (with a constant h).

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